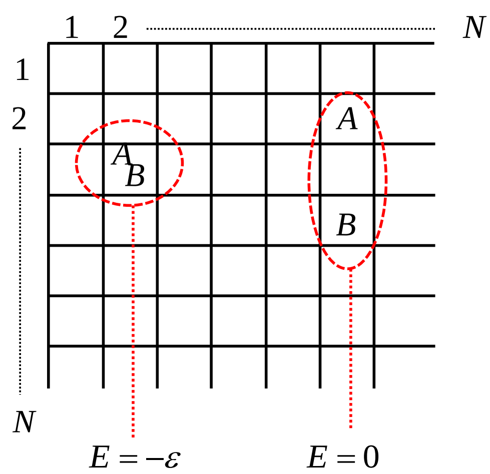
Resit
Second part (T4-T6) of
Thermal Physics
2020-2021

Saturday April 10, 2021
9.00-12.00
online

PROBLEM 1

Score: a+b+c+d+e=5+4+4+5+5=23

A 2-dimensional lattice is in equilibrium with a heath bath at temperature T. The lattice has $N \times N$ lattice positions (see figure). On this lattice there are two distinguishable particles A and B that can freely move over the lattice positions. In the situation that the particles are at the same lattice position the energy of the system of two particles is $-\varepsilon$; in the situation that the particles are at different lattice positions this energy is zero.



a) How many possibilities (microstates) are there to place the two particles on the lattice? How many of these microstates have energy $-\varepsilon$ and how many have energy zero.

b) Give the general expression of the partition function Z when you sum over all different *energies* of a system. Use this to show that the partition function Z for the system of the two particles on the lattice is:

$$Z = N^2\left(N^2 - 1 + e^{\beta\varepsilon}\right)$$

c) Use this partition function to calculate the internal energy U of the system of the two particles on the lattice.

d) Calculate the probability P_{same} that the particles are at the same lattice position.

e) Make a sketch of P_{same} as a function of temperature T. Clearly indicate the values of P_{same} at $T = 0$ and $T = \infty$.

PROBLEM 2

Score: a+b+c+d =6+6+5+5=22

Consider a 3D metallic crystal of hydrogen atoms, suppose that each atom can be in one of the following states:

State	N_e	Energy
Ground	1	$-\Delta$
Positive ion	0	$-\frac{1}{2}\Delta$
Negative ion	2	$\frac{1}{2}\Delta$
Excited	1	Δ

with N_e the number of electrons of the atom.

This crystal is in contact with heat bath at temperature T and an electron reservoir that can be characterized with a chemical potential μ. Each atom can be considered as a system that has four states.

a) Show that the grand partition function of one of these atoms can be written as:

$$\mathcal{z} = 2e^{\beta\mu}\left(\cosh(\beta\Delta) + \cosh\beta\left(\frac{1}{2}\Delta - \mu\right)\right)$$

b) Show that when the mean number $\langle N_e \rangle$ of electrons of an atom is equal to 1 then we have $\mu = \frac{1}{2}\Delta$.

c) Calculate the internal energy U of a system of N of these hydrogen atoms when $\mu = \frac{1}{2}\Delta$. Express your answer using $\sinh(\beta\Delta)$ and $\cosh(\beta\Delta)$. Show that $\frac{U}{N} \to 0$ in the high temperature limit and that $\frac{U}{N} \to -\Delta$ in the low temperature limit.

d) Explain how you would calculate the entropy of a system of N of these hydrogen atoms under the condition that $\mu = \frac{1}{2}\Delta$. Meaning, describe in words and formulas which steps you would take. You do not have to simplify the formulas.

PROBLEM 3

Score: a+b+c+d=6+6+5+5=22

The Berthelot equation of state is given by:

$$P = \frac{RT}{V-b} - \frac{a}{TV^2}$$

in which P, V, T are the pressure, the *molar* volume and the temperature of the gas, respectively. The constant a controls the attractive molecular interactions and the constant b corrects for the volume of the gas molecules.

a) Show that for a gas described by the Berthelot equation (a Berthelot gas), the critical temperature, pressure and volume are given by:

$$(T_c, P_c, V_c) = (\frac{2}{3}\sqrt{\frac{2}{3}\frac{a}{bR}}, \frac{1}{12b}\sqrt{\frac{2}{3}\frac{aR}{b}}, 3b)$$

b) Calculate the second virial coefficient $B(T)$ of the Berthelot gas.

c) Calculate the Boyle temperature T_b of the Berthelot gas.

A crude model for the intermolecular potential is the square well potential. Suppose that for a certain real gas (not necessarily a Berthelot gas) we have the following square well potential $v_{\kappa,R,\varepsilon}(r)$ describing the interaction between two molecules as a function of their separation distance r:

$$v(r) = \infty; \qquad 0 < r \leq \frac{R}{\kappa}$$

$$v(r) = -\varepsilon; \qquad \frac{R}{\kappa} < r \leq R$$

$$v(r) = 0; \qquad r > R$$

with κ a dimensionless constant such that $\kappa > 1$, ε has the units of energy and the radius R is expressed in units of length.

d) Calculate the second virial coefficient $B(T)$ (per mole) for a real gas with such a square well potential $v_{\kappa,R,\varepsilon}(r)$. Express your answer in terms of R, ε, κ and β.

PROBLEM 4
Score: a+b+c+d =7+6+5+5=23

Consider a one-atom layer thick square (with sides of length L) of metallic atoms. The square consists of N atoms of which each atom contributes exactly *two* electrons to the total amount of conduction electrons. These conduction electrons can be considered as a 2D ideal gas of fermions with spin $\frac{1}{2}$ enclosed in a square with area $A = L^2$.

a) Show that number of states $\Gamma(E)$ with energy smaller than E for this 2D gas of fermions is proportional to E:

$$\Gamma(E) = \frac{AE}{\sigma}$$

with σ a constant. Give an expression for σ in terms of fundamental constants.

Use the expression for $\Gamma(E)$ to show that the density of states $g(E)dE$ for this 2D ideal gas of electrons is independent of energy and can be written as:

$$g(E)dE = \frac{AdE}{\sigma}$$

We now cool the square of metallic atoms to temperature $T = 0$.

b) Calculate the Fermi energy E_F for this 2D ideal gas of electrons. Express your answer in N, A and σ.
c) Show that at $T = 0$ the internal energy of this gas is given by $U = NE_F$.
d) Calculate the 2D pressure (surface tension) of this gas at $T = 0$.

Solutions

PROBLEM 1

a)

Total number of microstates: $N \times N$ possibilities for the first particle times $N \times N$ possibilities for the second particle gives N^4 microstates. The number of microstates with energy $E = -\varepsilon$ is equal to the number of lattice positions: N^2. The number of microstates that have energy $E = 0$ is $N^4 - N^2 = N^2(N^2 - 1)$.

b)

General expression for the partition function:

$$Z = \sum_r e^{-\beta E_r}$$

where the summation is over all microstates r or

$$Z = \sum_{E_r} g(E_r) e^{-\beta E_r}$$

where the summation is over all different energies E_r and $g(E_r)$ is the degeneracy of the energy E_r (number of microstates with that energy).

For the system of the two particles on the lattice:

$$Z = N^2 e^{-\beta(-\varepsilon)} + N^2(N^2 - 1)e^{-\beta 0} = N^2(N^2 - 1) + N^2 e^{\beta\varepsilon} = N^2\left(N^2 - 1 + e^{\beta\varepsilon}\right)$$

c)

We use $U = -\frac{\partial \ln Z}{\partial \beta}$ and find,

$$U = -\frac{N^2 \varepsilon e^{\beta\varepsilon}}{N^2(N^2 - 1) + N^2 e^{\beta\varepsilon}} = \frac{-\varepsilon e^{\beta\varepsilon}}{(N^2 - 1) + e^{\beta\varepsilon}}$$

d)

$$P_{same} = \frac{N^2 e^{\beta\varepsilon}}{Z} = \frac{N^2 e^{\beta\varepsilon}}{N^2(N^2 - 1) + N^2 e^{\beta\varepsilon}} = \frac{e^{\beta\varepsilon}}{(N^2 - 1) + e^{\beta\varepsilon}}$$

e)

If $T \to 0$ then $\beta \to \infty$ and thus:

$$P_{same} = \frac{e^{\beta\varepsilon}}{(N^2 - 1) + e^{\beta\varepsilon}} = \frac{1}{(N^2 - 1)e^{-\beta\varepsilon} + 1} \xrightarrow[\beta\to\infty]{} 1$$

If $T \to \infty$ then $\beta \to 0$ and thus:

$$P_{same} = \frac{e^{\beta\varepsilon}}{(N^2 - 1) + e^{\beta\varepsilon}} \xrightarrow[\beta\to\infty]{} \frac{1}{(N^2 - 1) + 1} = \frac{1}{N^2}$$

PROBLEM 2

a)

$$
\begin{aligned}
z = \sum_i e^{\beta(N_i\mu - E_i)} &= e^{\beta(\mu+\Delta)} + e^{\beta\frac{1}{2}\Delta} + e^{\beta(2\mu-\frac{1}{2}\Delta)} + e^{\beta(\mu-\Delta)} \\
&= e^{\beta\mu}\left(e^{\beta\Delta} + e^{-\beta\Delta}\right) + e^{\beta\mu}\left(e^{\beta\left(\frac{1}{2}\Delta-\mu\right)} + e^{-\beta\left(\frac{1}{2}\Delta-\mu\right)}\right) \\
&= 2e^{\beta\mu}\cosh(\beta\Delta) + 2e^{\beta\mu}\cosh\left(\frac{1}{2}\Delta-\mu\right) \\
&= 2e^{\beta\mu}\left(\cosh(\beta\Delta) + \cosh\beta\left(\frac{1}{2}\Delta-\mu\right)\right)
\end{aligned}
$$

in which the four terms in the sum represent the ground, positive ion, negative ion and excited state, respectively.

b)

The mean number of electrons $\langle N_e \rangle$ follows from,

$$
\begin{aligned}
\langle N_e \rangle = \sum_i P_i N_i &= \frac{e^{\beta(\mu+\Delta)}}{z} \times 1 + \frac{e^{\beta\frac{1}{2}\Delta}}{z} \times 0 + \frac{e^{\beta\left(2\mu-\frac{1}{2}\Delta\right)}}{z} \times 2 + \frac{e^{\beta(\mu-\Delta)}}{z} \times 1 = \\
&= \frac{e^{\beta(\mu+\Delta)} + 2e^{\beta\left(2\mu-\frac{1}{2}\Delta\right)} + e^{\beta(\mu-\Delta)}}{e^{\beta(\mu+\Delta)} + e^{\beta\frac{1}{2}\Delta} + e^{\beta\left(2\mu-\frac{1}{2}\Delta\right)} + e^{\beta(\mu-\Delta)}}
\end{aligned}
$$

Equating the last expression to 1.

$$
\begin{aligned}
\langle N_e \rangle = 1 &\Rightarrow e^{\beta(\mu+\Delta)} + 2e^{\beta(2\mu-\frac{1}{2}\Delta)} + e^{\beta(\mu-\Delta)} = e^{\beta(\mu+\Delta)} + e^{\beta\frac{1}{2}\Delta} + e^{\beta(2\mu-\frac{1}{2}\delta)} + e^{\beta(\mu-\Delta)} \\
&\Rightarrow e^{\beta(2\mu-\frac{1}{2}\Delta)} = e^{\beta\frac{1}{2}\Delta} \Rightarrow e^{2\beta\mu} = e^{\beta\Delta} \Rightarrow \mu = \frac{1}{2}\Delta
\end{aligned}
$$

The mean number of electrons can be further written (not necessary) as:

$$
\begin{aligned}
\langle N_e \rangle &= \frac{e^{\beta\mu}\left(e^{\beta\Delta} + e^{-\beta\Delta} + 2e^{\beta(\mu-\frac{1}{2}\Delta)}\right)}{2e^{\beta\mu}\left(\cosh(\beta\Delta) + \cosh\beta\left(\mu-\frac{1}{2}\Delta\right)\right)} = \frac{2e^{\beta\mu}\left(\cosh(\beta\Delta) + e^{\beta(\mu-\frac{1}{2}\Delta)}\right)}{2e^{\beta\mu}\left(\cosh(\beta\Delta) + \cosh\beta\left(\mu-\frac{1}{2}\Delta\right)\right)} \\
&= \frac{\left(\cosh(\beta\Delta) + e^{\beta(\mu-\frac{1}{2}\Delta)}\right)}{\left(\cosh(\beta\Delta) + \cosh\beta\left(\mu-\frac{1}{2}\Delta\right)\right)}
\end{aligned}
$$

Equating this to 1 gives:

$$e^{\beta(\mu-\frac{1}{2}\Delta)} = \frac{1}{2}e^{\beta(\mu-\frac{1}{2}\Delta)} + \frac{1}{2}e^{-\beta(\mu-\frac{1}{2}\Delta)} \Rightarrow e^{\beta(\mu-\frac{1}{2}\Delta)} = e^{-\beta(\mu-\frac{1}{2}\Delta)} \Rightarrow \mu - \frac{1}{2}\Delta = -\left(\mu - \frac{1}{2}\Delta\right) \Rightarrow \mu = \frac{1}{2}\Delta$$

c)

The internal energy for a system of N hydrogen atoms is

$$U = N\langle E\rangle$$

with

$$\langle E\rangle = \sum_i P_i E_i = \frac{e^{\beta(\mu+\Delta)}}{\mathcal{Z}} \times (-\Delta) + \frac{e^{\beta\frac{1}{2}\Delta}}{\mathcal{Z}} \times \left(-\frac{1}{2}\Delta\right) + \frac{e^{\beta(2\mu-\frac{1}{2}\Delta)}}{\mathcal{Z}} \times \left(\frac{1}{2}\Delta\right) + \frac{e^{\beta(\mu-\Delta)}}{\mathcal{Z}} \times (\Delta)$$
$$= \frac{-\Delta}{\mathcal{Z}}\left[e^{\beta\mu}\left(e^{\beta\Delta} - e^{-\beta\Delta}\right) + e^{\beta\mu}\left(\frac{e^{\beta\left(\frac{1}{2}\Delta-\mu\right)} - e^{-\beta\left(\frac{1}{2}\Delta-\mu\right)}}{2}\right)\right]$$
$$= \frac{-\Delta}{\mathcal{Z}}\left[2e^{\beta\mu}\sinh(\beta\Delta) + e^{\beta\mu}\sinh\left(\beta\left(\frac{1}{2}\Delta - \mu\right)\right)\right]$$

Which can be further simplified to,

$$\langle E\rangle = -\Delta\frac{2e^{\beta\mu}\sinh(\beta\Delta) + e^{\beta\mu}\sinh\left(\beta\left(\frac{1}{2}\Delta - \mu\right)\right)}{2e^{\beta\mu}\left(\cosh(\beta\Delta) + \cosh\beta\left(\frac{1}{2}\Delta - \mu\right)\right)} = -\Delta\frac{\sinh(\beta\Delta) + \frac{1}{2}\sinh\left(\beta\left(\frac{1}{2}\Delta - \mu\right)\right)}{\left(\cosh(\beta\Delta) + \cosh\beta\left(\frac{1}{2}\Delta - \mu\right)\right)}$$

In case we have $\mu = \frac{1}{2}\Delta$ we have,

$$\langle E\rangle = -\Delta\frac{\sinh(\beta\Delta)}{(\cosh(\beta\Delta) + 1)}$$

It is also OK to do the substitution $\mu = \frac{1}{2}\Delta$ directly then we have

$$\langle E\rangle = \sum_i P_i E_i = \frac{e^{\frac{3}{2}\beta\Delta}}{\mathcal{Z}} \times (-\Delta) + \frac{e^{\frac{1}{2}\beta\Delta}}{\mathcal{Z}} \times \left(-\frac{1}{2}\Delta\right) + \frac{e^{\frac{1}{2}\beta\Delta}}{\mathcal{Z}} \times \left(\frac{1}{2}\Delta\right) + \frac{e^{-\frac{1}{2}\beta\Delta}}{\mathcal{Z}}(\Delta)$$

With

$$\mathcal{Z} = 2e^{\beta\mu}(\cosh(\beta\Delta) + \cosh\beta(0)) = 2e^{\frac{1}{2}\beta\Delta}(\cosh(\beta\Delta) + 1)$$

This becomes,

$$\langle E\rangle = \frac{e^{\beta\Delta}}{2(\cosh(\beta\Delta)+1)}\times(-\Delta) + \frac{e^{-\beta\Delta}}{2(\cosh(\beta\Delta)+1)}(\Delta) = \langle E\rangle = -\Delta\frac{\sinh(\beta\Delta)}{(\cosh(\beta\Delta)+1)}$$

And thus,

$$U = N\langle E\rangle = -N\Delta\frac{\sinh(\beta\Delta)}{(\cosh(\beta\Delta)+1)}$$

In the high temperature limit, we have:

$$\langle E\rangle = -\Delta\frac{\sinh(\beta\Delta)}{(\cosh(\beta\Delta)+1)} \xrightarrow[\beta\to 0]{} = -\Delta\frac{0}{(1+1)} = 0$$

In the low temperature limit we have:

$$\langle E\rangle = -\Delta\frac{\sinh(\beta\Delta)}{(\cosh(\beta\Delta)+1)} = -\Delta\frac{\tanh(\beta\Delta)}{\left(1+\frac{1}{\cosh(\beta\Delta)}\right)} \xrightarrow[\beta\to\infty]{} = -\Delta\frac{1}{(1+0)} = -\Delta$$

d)

Use Blundell and Blundell equation 22.23, and calculate the entropy S_1 for 1 atom,

$$S_1 = \frac{\langle E\rangle - \mu\langle N_e\rangle + kT\ln \mathcal{Z}}{T} = \frac{-\Delta\frac{\sinh(\beta\Delta)}{(\cosh(\beta\Delta)+1)} - \mu + kT\ln\left(2e^{\beta\mu}(\cosh(\beta\Delta)+1)\right)}{T}$$

$$= \frac{-\Delta\frac{\sinh(\beta\Delta)}{(\cosh(\beta\Delta)+1)} - \frac{1}{2}\Delta\ + kT\ln\left(2e^{\frac{1}{2}\beta\Delta}(\cosh(\beta\Delta)+1)\right)}{T}$$

For N atoms we have $S = NS_1$

PROBLEM 3
a)
The critical point is found when the isotherm of the gas has an infliction point, thus as,

$$\left(\frac{\partial P}{\partial V}\right)_T = \left(\frac{\partial^2 P}{\partial V^2}\right)_T = 0$$

This gives (together with the Berthelot equation of state) three equations with three unknowns namely:

$$P = \frac{RT}{V-b} - \frac{a}{TV^2} \qquad \text{eq(1)}$$

$$\left(\frac{\partial P}{\partial V}\right)_T = \frac{-RT}{(V-b)^2} + \frac{2a}{TV^3}$$

Thus $\left(\frac{\partial P}{\partial V}\right)_T = 0$ leads to

$$\left(\frac{-RT}{(V-b)^2} + \frac{2a}{TV^3}\right) = 0 \Rightarrow \frac{2RT^2}{(V-b)^2} = \frac{4a}{V^3} \qquad \text{eq(2)}$$

$$\left(\frac{\partial^2 P}{\partial V^2}\right)_T = \left(\frac{2RT}{(V-b)^3} - \frac{6a}{TV^4}\right)$$

And $\left(\frac{\partial^2 P}{\partial V^2}\right)_T = 0$ leads to

$$\left(\frac{2RT}{(V-b)^3} - \frac{6a}{TV^4}\right) = 0 \Rightarrow \frac{2RT^2}{(V-b)^2} = \frac{6a(V-b)}{V^4} \qquad \text{eq(3)}$$

Combining 2 and 3 leads to

$$\frac{4a}{V^3} = \frac{6a(V-b)}{V^4} \Rightarrow 4V = 6(V-b) \Rightarrow V_c = 3b$$

Substituting this in equation 2 gives,

$$\frac{2RT^2}{(3b-b)^2} = \frac{4a}{(3b)^3} = 0 \Rightarrow \frac{2RT^2}{4b^2} = \frac{4a}{27b^3} \Rightarrow T_c = \frac{2}{3}\sqrt{\frac{2}{3}\frac{a}{bR}}$$

and $T_c = \frac{2}{3}\sqrt{\frac{2}{3}\frac{a}{bR}}$ and $V_c = 3b$ in equation 1 gives,

$$P_c = \frac{R\frac{2}{3}\sqrt{\frac{2}{3}\frac{a}{bR}}}{3b-b} - \frac{a}{\frac{2}{3}\sqrt{\frac{2}{3}\frac{a}{bR}}(3b)^2} = \frac{1}{3b}\sqrt{\frac{2}{3}\frac{aR}{b}} - \frac{3a}{18b^2}\sqrt{\frac{3}{2}\frac{bR}{a}}$$

$$= \frac{1}{3b}\sqrt{\frac{2}{3}\frac{aR}{b}} - \frac{9}{36b}\sqrt{\frac{2}{3}\frac{aR}{b}} = \frac{1}{12b}\sqrt{\frac{2}{3}\frac{aR}{b}}$$

Consequently,

$$(T_c, P_c, V_c) = (\frac{2}{3}\sqrt{\frac{2}{3}\frac{a}{bR}}, \frac{1}{12b}\sqrt{\frac{2}{3}\frac{aR}{b}}, 3b)$$

b)
Rewrite the Berthelot equation as:

$$\frac{PV}{RT} = \frac{1}{\left(1-\frac{b}{V}\right)} - \frac{a}{RT^2V}$$

and expand the first term on the right-hand side in powers of $\frac{1}{V}$:

$$\frac{PV}{RT} = \left(1 + \frac{b}{V} + \left(\frac{b}{V}\right)^2 + \cdots\right) - \frac{a}{RT^2V} \Rightarrow$$

$$\frac{PV}{RT} = 1 + \left(b - \frac{a}{RT^2}\right)\frac{1}{V} + \cdots$$

Thus,

$$B(T) = b - \frac{a}{RT^2}$$

c)
The temperature at which the second virial coefficient is zero is called the Boyle temperature.

$$B(T) = 0 \Rightarrow b - \frac{a}{RT^2} = 0 \Rightarrow T_b = \sqrt{\frac{a}{bR}}$$

At this temperature Boyle's law $(PV = constant)$ approximately holds for a real gas.

d)

$$B(T) = \frac{N}{2}\int\left(1 - e^{-\beta v(r)}\right)d^3r$$

$$= \frac{N}{2}\int_0^{\frac{R}{\kappa}} 4\pi r^2 dr + \frac{N}{2}\int_{\frac{R}{\kappa}}^{R}\left(1 - e^{\beta\varepsilon}\right)4\pi r^2 dr + \frac{N}{2}\int_R^{\infty} 0\cdot 4\pi r^2 dr \Rightarrow$$

$$\frac{B(T)}{N} = 2\pi\int_0^{\frac{R}{\kappa}} r^2 dr + 2\pi\left(1 - e^{\beta\varepsilon}\right)\int_{\frac{R}{\kappa}}^{R} r^2 dr + 0$$

$$= \frac{2\pi}{3}\left(\left(\frac{R}{\kappa}\right)^3 + \left(1 - e^{\beta\varepsilon}\right)\left(R^3 - \left(\frac{R}{\kappa}\right)^3\right)\right) \Rightarrow$$

$$\frac{B(T)}{N} = \frac{2\pi}{3}\left(R^3 - e^{\beta\varepsilon}\left(R^3 - \left(\frac{R}{\kappa}\right)^3\right)\right) = \frac{2\pi}{3}R^3\left(1 - e^{\beta\varepsilon}\left(\frac{\kappa^3 - 1}{\kappa^3}\right)\right)$$

PROBLEM 4

a)

From the solution of the 2D-wave equation: $\varphi = A \sin k_x x \sin k_y y$ and taking this function to vanish at $x = y = 0$ and at $x = y = L$ results in,

$k_x = \frac{n_x \pi}{L}$ and $k_y = \frac{n_y \pi}{L}$ with n_x and n_y non-zero positive integers.

The total number of states with $|\vec{k}| < k$ is then given by, (the area of a quarter circle because we have only positive integers, with radius k divided by the area of the unit surface e.g. the surface of one state, in k-space).

$$\Gamma(k) = \frac{\frac{1}{4}\pi k^2}{\left(\frac{\pi}{L}\right)^2} = \frac{1}{4}\frac{L^2 k^2}{\pi}$$

Converting to energy $p = \sqrt{2mE} = \hbar k$ we find, $k = \frac{\sqrt{2mE}}{\hbar}$ and $dk = \frac{1}{2}\frac{2m}{\hbar\sqrt{2mE}}dE$

We find,

$$\Gamma(E) = \frac{1}{4}\frac{L^2 k^2}{\pi} = \frac{1}{4}\frac{L^2}{\pi}\frac{2mE}{\hbar^2} = \frac{1}{2}\frac{A}{\pi}\frac{mE}{\hbar^2}$$

For fermions with spin $\frac{1}{2}$ we have two spin states thus,

$$\Gamma(E) = 2 \times \frac{1}{2}\frac{A}{\pi}\frac{mE}{\hbar^2} = \frac{A}{\pi}\frac{mE}{\hbar^2} = \frac{AE}{\sigma}$$

Thus, $\sigma = \frac{\pi\hbar^2}{m}$

The number of states between $E + dE$ and E is:

$$g(E)dE = \Gamma(E + dE) - \Gamma(E) = \frac{\partial \Gamma}{\partial E}dE = \frac{AdE}{\sigma}$$

b)

The Fermi energy is the value of the chemical potential μ at absolute zero temperature:

$$E_F = \mu(T = 0)$$

Total number of fermions is given by,

$$2N = \int_0^\infty n(E)\, g(E) dE$$

with,

$$n(E) = \frac{1}{e^{\beta(E-\mu)} + 1}$$

the mean number of fermions with energy E (Fermi-Dirac distribution)

Thus,

$$2N = \frac{A}{\sigma}\int_0^\infty \frac{dE}{e^{\beta(E-\mu)} + 1}$$

And at $T = 0$, we have $E_F = \mu(T = 0)$ and thus $n(E) = 1$ if $E < E_F$ and $n(E) = 0$ if $E > E_F$. Thus,

$$2N = \frac{A}{\sigma}\int_0^{E_F} dE = \frac{A}{\sigma} E_F \Rightarrow E_F = \frac{2N\sigma}{A}$$

c)

$$U = \int_0^\infty En(E)\, g(E) dE = \frac{A}{\sigma}\int_0^\infty \frac{EdE}{e^{\beta(E-\mu)} + 1} \underset{T=0}{\Longrightarrow} U = \frac{A}{\sigma}\int_0^{E_F} EdE = \frac{A}{2\sigma} E_F^2 = NE_F$$

d)

For a 2D system with a fixed number of particles we have ($dN = 0$ and $dV = dA$)

$$dU = TdS - PdA \underset{T\to 0}{\Longrightarrow} P = -\frac{dU}{dV}$$

Using

$$U = NE_F = \frac{2N^2\sigma}{A}$$

We find,

$$P = \frac{2N^2}{A^2}\sigma = \frac{NE_F}{A}$$